The Stability of Couette Flow Between Rotating Cylinders in the Presence of a Radial Temperature Gradient

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The effects of viscosity and density variations due to an imposed radial temperature gradient on the stability of Couette flow between rotating cylinders are investigated. The annular spacing between the cylinders is assumed to be small compared with the mean radius. The fluids considered are water and 50% aqueous glycerol. Free convection due to gravity is not considered.

Approximate solutions to the stability equations are obtained by the Galerkin method. Computations are restricted to the case where the outer cylinder is at rest. For the cases studied, the effects of radial convection were found to be small but the effects of the temperature dependence of viscosity were appreciable. The critical Taylor number based on the mean temperature viscosity was found to decrease as the viscosity variation became more pronounced and as the Prandtl number increased.

The motion of a viscous fluid between concentric rotating cylinders becomes unstable when dynamic conditions are reached where it becomes impossible for the radial pressure gradient and viscous forces to dampen and restore changes in centrifugal force caused by small disturbances to the flow. This instability usually results in a secondary flow in the form of steady, counter-rotating toroidal vortices. The critical dynamic conditions at which instability occurs were first investigated theoretically and experimentally by Taylor (*1*) in 1923. Taylor formulated the stability problem, developed an elaborate method for solving it, and obtained an approximate formula for predicting the onset of instability.

Subsequent analyses have been oriented toward obtaining suitable mathematical techniques to simplify the solution to the stability problem and to extend computations to a wider range of parameters than that covered by Taylor. The earlier analyses appearing in the literature are limited to cases where the annular spacing is small compared with the mean cylinder radius (small-gap approximations). A comprehensive solution to the small-gap stability problem was obtained by Chandrasekhar (*2*) in 1954. Recently, mathematical techniques have been developed which have facilitated solutions to the stability equations without the use of simplifying small-gap approximations. Solutions to various finite-gap stability problems are given in references *3* to *5*.

In general, theoretical solutions to the Taylor problem have been found to be in excellent agreement with experimental results. Measurements of the conditions at the onset of instability for various fluids and cylinder geometries are given in references *1* and *6* to *8*.

The problem of heat transfer to rotating cylinders has been of considerable engineering interest and has many applications to rotating machinery. For many of these cases the outer cylinder is at rest and the narrow gap approximation is applicable. A characteristic parameter associated with this configuration is the Taylor number

$$N_{Ta} = \frac{\Omega_i^2 d^3 R_c}{\nu^2}$$

In general, the secondary vortex flow has been found to set in when N_{Ta} exceeds a certain critical value $N_{Ta_{cr}}$.

Most of the investigations of heat transfer to the vortex flow have been experimental in nature, the results of several are given in references *9* to *12*. In general the correlations obtained in these investigations depend on the value of $N_{Ta_{cr}}$. Thus it is important to understand the factors governing critical conditions for analyzing heat transfer in the vortex regime, as well as determining when the vortex flow will set in.

The effects of a radial temperature gradient on stability are due to property variations with temperature. Density variations affect the flow in two ways: a radial convective effect due to the interaction of a density gradient with centrifugal force, and free convection due to the interaction of the density gradient with gravity. The temperature dependence of viscosity is of inherent importance due to the appearance of the kinematic viscosity in the Taylor number. If one were to attempt to use the isothermal stability criteria $N_{Ta} = N_{Ta_{cr}}$, to determine the critical inner cylinder angular velocity when a temperature gradient is imposed, the problem of choosing the correct kinematic viscosity would arise.

Previous theoretical analyses of the stability of Couette flow between rotating cylinders in the presence of a radial temperature gradient have deviated from isothermal theory only by the inclusion of the effect of radial convection. Solutions to this problem with the use of small-gap approximations and various approximations to the primary velocity profile are given in references *13* and *14*. The solution to this small-gap stability problem without further approximation to the velocity profile was obtained by Lai (*15*) in 1962. Finally, the solution to this problem without the use of small-gap approximations may be found in a recent paper by Walowit, Tsao, and DiPrima (*5*).

The following analysis, which is simplified by the use of small-gap approximations, includes both the effects of radial convection and variable viscosity. Due to the large number parameters that arise, computations are restricted to the case where the outer cylinder is at rest. The fluids analyzed are water and a 50% water-glycerol solution.

Results are restricted to relatively small temperature differences (40°C. or less) for which case thermal diffusivities are relatively constant; thus temperature variations in thermal diffusivity are neglected. Convection under the influence of gravity is not considered.

FORMULATION OF THE STABILITY PROBLEM

The stability problem will be formulated by considering the behavior of small rotationally symmetric disturbances to primary velocity and temperature profiles in a manner similar to that employed by Taylor. If these disturbances are damped out, the flow is said to be stable. If they amplify, then the flow is unstable.

The governing equations are the momentum, continuity, and energy equations for rotationally symmetric flow. The terms involving viscous dissipation and the energy associated with compression are neglected in the energy equation. The fluid density is taken to be constant except for the term involving the interaction of the density gradient with centripetal acceleration. Here, the equation of state

$$\rho = \rho_c \left[1 - \alpha \left(T - T_c\right)\right] \tag{1}$$

is used. This is the well-known Boussinesq approximation.

The resulting equations have a steady solution for which the radial and axial velocity components are zero. The tangential angular velocity profile and the temperature profile are given by

$$\overline{V}/r = \Omega = \Omega_i + (\Omega_o - \Omega_i) \left(\int_{R_i}^{r} \frac{d\xi}{\mu \xi^3}\right) \Big/ \left(\int_{R_i}^{R_o} \frac{d\xi}{\mu \xi^3}\right) \tag{2}$$

and

$$\overline{T} = T_i + (T_o - T_i) \frac{\ln (r/R_i)}{\ln (R_o/R_i)} \tag{3}$$

The equations for Ω and $\overline{T}$ represent the primary velocity and temperature profiles. In the usual manner of hydrodynamic stability theory, one now proceeds by considering the primary flow to be perturbed by small disturbances dependent on r, z, and t. Thus the variables appearing in the governing equations will be expressed in the following form:

$$\begin{aligned} u_r &= u' \\ u_\theta &= \overline{V} + v' \\ u_z &= w' \\ T &= \overline{T} + T' \\ p &= \overline{p} + p' \\ \rho &= \overline{\rho} + \rho' = \overline{\rho} - \rho_c \alpha T' \\ \mu &= \overline{\mu} + \mu' = \overline{\mu} + \frac{\partial \overline{\mu}}{\partial \overline{T}} T' \end{aligned} \tag{4}$$

The barred quantities are dependent on r alone and the primed quantities represent small disturbances dependent on r, z, t.

The linear disturbance equations are obtained by substituting the above equations for the respective variables in the momentum, energy, and continuity equations and by neglecting quadratic terms in the primed quantities. The following system of linear partial differential equations is then obtained:

$$\rho_c \left(\frac{\partial u'}{\partial t} + 2 \frac{\overline{V}}{r} v'\right) - \frac{\overline{V}^2}{r} \rho' = - \frac{\partial p'}{\partial r} + \overline{\mu} \left(\Delta - \frac{1}{r^2}\right) u' + 2 \frac{d\overline{\mu}}{dr} \frac{\partial u'}{\partial r} \tag{5}$$

$$\rho_c \left[\frac{\partial v'}{\partial t} + \left(\frac{d\overline{V}}{dr} + \frac{\overline{V}}{r}\right) u'\right] = \left(\Delta \overline{V} - \frac{\overline{V}}{r^2}\right) \mu' + \overline{\mu} \left(\Delta - \frac{1}{r^2}\right) v' + r \frac{d}{dr} \left(\frac{\overline{V}}{r}\right) \frac{\partial \mu'}{\partial r} + \frac{d\overline{\mu}}{dr} r \frac{\partial}{\partial r} \left(\frac{v'}{r}\right)$$

$$\rho_c \frac{\partial w'}{\partial t} = - \frac{\partial p'}{\partial z} + \overline{\mu} \Delta w' + \frac{d\overline{\mu}}{dr} \frac{\partial w'}{\partial r} + \frac{d\overline{\mu}}{dr} \frac{\partial u'}{\partial z} \tag{7}$$

$$\frac{\partial u'}{\partial r} + \frac{u'}{r} + \frac{\partial w'}{\partial z} = 0 \tag{8}$$

$$\frac{\partial T'}{\partial t} + \frac{d\overline{T}}{dr} u' = \kappa \Delta T' \tag{9}$$

The differential operator Δ is defined by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{10}$$

Since the imposed boundary conditions already have been satisfied by the primary velocity and temperature profiles, the disturbance velocities and temperatures will be zero at the cylinder walls:

$$u' = v' = w' = T' = 0, \text{ at } r = R_i, R_o \tag{11}$$

For a given viscosity temperature relationship, the five dependent variables in Equations (5) to (9) are u', v', w', p', and T'. These disturbance quantities may now be expanded in a manner similar to that used by Taylor. Since the coefficients in Equations (5) to (9) depend only on r, one may attempt to separate variables and look for solutions in the form

$$\begin{aligned} u' &= u^* (r)\, e^{\sigma t} \cos \lambda z \\ v' &= v^* (r)\, e^{\sigma t} \cos \lambda z \\ w' &= w^* (r)\, e^{\sigma t} \sin \lambda z \\ T' &= \Theta^* (r)\, e^{\sigma t} \cos \lambda z \\ p' &= \rho_c \tau (r)\, e^{\sigma t} \cos \lambda z \end{aligned} \tag{12}$$

When the relationships given by Equation (12) are substituted for the disturbance quantities appearing in Equations (5) to (11), variables do separate and a system of five linear ordinary differential equations results. These equations contain the additional parameters λ (the wave number of the disturbance) and σ.

In general, σ can be complex. The stability of the flow will thus depend upon the sign of the real part of σ. If $Re(\sigma)$ is positive, disturbances will amplify; conversely if $Re(\sigma)$ is negative, disturbance will be damped out. Since the situation of interest corresponds to the marginal case where disturbances begin to amplify, solutions will be sought for which $Re(\sigma) = 0$. In addition, since the secondary flow has been observed to be nonoscillatory in nature, solutions will be restricted to cases where both the real and imaginary parts of σ are equal to zero.

The radially dependent disturbance equations and the primary flow quantities may be simplified further with the use of the well-known small-gap approximations. These approximations consist of neglecting terms of order $1/r$ compared with $\partial/\partial r$, and d/r compared with 1 and of retaining terms of order λd and $d^3 r^2 \Omega^2/\nu^2$.

In accordance with approximations of this type, the primary temperature profile given by Equation (3) reduces to the linear conduction profile

$$\overline{T} = T_c + (T_o - T_i)\, x \tag{13}$$

where T_c is the mean temperature and $x = (r - R_c)/d$ is a stretched position coordinate.

The viscosity variations of many liquids over short temperature ranges may be described empirically by the relationship

$$\mu = \frac{1}{K_1 T^2 + K_2 T + K_3} \quad (14)$$

In the present study these constants are determined by requiring the viscosity function given in Equation (14) to fit experimental data at temperatures T_i, T_o, and T_c. The primary viscosity distribution may thus be written as a function of x as follows:

$$\tilde{\mu} = \frac{\bar{\mu}}{\mu_c} = \frac{1}{Ax^2 + Bx + 1} \quad (15)$$

where

$$A = 2\left(\frac{1}{\tilde{\mu}_i} + \frac{1}{\tilde{\mu}_o}\right) - 4$$

and

$$B = \frac{1}{\tilde{\mu}_o} - \frac{1}{\tilde{\mu}_i}$$

The corresponding angular velocity profile may now be written as

$$\tilde{\Omega} = \frac{\Omega}{\Omega_c} = 1 + \gamma f(x) \quad (16)$$

where $f(x)$ is given by

$$f(x) = \frac{A}{3}x^3 + \frac{B}{2}x^2 + x \quad (17)$$

and

$$\gamma = \left(\frac{\Omega_o}{\Omega_c} - 1\right) \Big/ f\left(\frac{1}{2}\right) \quad (18)$$

The five radially dependent disturbance equations may be arranged to eliminate w^* and τ. When the small-gap approximations are employed along with the introduction of the dimensionless disturbance amplitudes and wave number

$$u = \frac{u^*}{R_c\Omega_c}, \; v = \frac{\nu_c}{(R_c\Omega_c)^2 \gamma d} v^*$$

$$\Theta = \frac{\kappa}{R_c\Omega_c\, d(T_o - T_i)} \Theta^*, \; a = \lambda d \quad (19)$$

the dimensionless stability equations take the form

$$\frac{d^2}{dx^2}\left(\tilde{\mu}\frac{d^2u}{dx^2}\right) - 2a^2\frac{d}{dx}\left(\tilde{\mu}\frac{du}{dx}\right) + \left(a^4\tilde{\mu} + a^2\frac{d^2\tilde{\mu}}{dx^2}\right)u = -a^2\tilde{N}_{Ta}\,(\tilde{\Omega}v + N\tilde{\Omega}^2\Theta) \quad (20)$$

$$\frac{d}{dx}\left(\tilde{\mu}\frac{dv}{dx}\right) - a^2\tilde{\mu}v + N_{Pr}\frac{d}{dx}\left(\frac{d\tilde{\mu}}{dx}\frac{df}{dx}\Theta\right) = \frac{df}{dx}u \quad (21)$$

$$\frac{d^2\Theta}{dx^2} - a^2\Theta = u \quad (22)$$

$$u = \frac{du}{dx} = v = \Theta = 0 \text{ at } x = \pm\frac{1}{2} \quad (23)$$

The parameters $\tilde{N}_{Ta}$, N_{Pr}, and N in the above equations are defined as follows:

$$\tilde{N}_{Ta} = -\frac{2d^3R_c\Omega_c^2\gamma}{\nu_c^2} \quad (24)$$

$$N_{Pr} = \frac{\nu_c}{\kappa} \quad (25)$$

$$N = -\frac{N_{Pr}\alpha(T_o - T_i)}{2\gamma} \quad (26)$$

The quantity $\tilde{N}_{Ta}$ is a modified Taylor number which reduces to N_{Ta} if the viscosity is taken to be independent of temperature. The parameter N characterizes the relative effect of radial convection on stability.

Equations (20) to (23) are similar to the equations governing the stability of the boundary layer over a curved surface which were formulated by DiPrima and Dunn (*16*). The principal differences are due to differences in the boundary conditions and the primary velocity profile. If viscosity variations are neglected, Equations (20) to (23) reduce to a form equivalent to those previously solved by Lai.

SOLUTION OF THE STABILITY PROBLEM

Equations (20) to (23) represent an eighth-order system with homogeneous boundary conditions. Thus an eigenvalue problem is presented for which solutions will exist only if the various parameters appearing in the system obey a definite secular relationship, which may be written in the form

$$F(a, \tilde{N}_{Ta}, \tilde{\mu}_i, \tilde{\mu}_o, \tilde{\Omega}_o, N_{Pr}, N) = 0 \quad (27)$$

For given values of $\tilde{\mu}_i$, $\tilde{\mu}_o$, $\tilde{\Omega}_o$, N_{Pr}, and N, the stability criterion is determined by finding the minimum value of $\tilde{N}_{Ta}$ overall positive a, which satisfies Equation (27).

Approximate solutions to the eigenvalue problem may be conveniently obtained by the Galerkin method. The functions of u, v, and Θ are expanded in complete sets of functions, satisfying the boundary conditions given by Equation (23). One may then write

$$u = \sum_{n=1}^{\infty} A_n u_n \quad (28)$$

$$v = \sum_{n=1}^{\infty} B_n v_n \quad (29)$$

$$\Theta = \sum_{n=1}^{\infty} C_n \Theta_n \quad (30)$$

These expansions may be substituted for u, v, and Θ in Equations (20) to (22) to yield

$$\sum_{n=1}^{\infty}\left\{A_n\left[\frac{d^2}{dx^2}\left(\tilde{\mu}\frac{d^2u_n}{dx^2}\right) - 2a^2\frac{d}{dx}\left(\tilde{\mu}\frac{du_n}{dx}\right) + \left(a^4\tilde{\mu} + a^2\frac{d^2\tilde{\mu}}{dx^2}\right)u_n\right] + a^2\tilde{N}_{Ta}(B_n\tilde{\Omega}v_n + C_nN\tilde{\Omega}^2\Theta_n)\right\} = \epsilon^{(1)}(x) \quad (31)$$

$$\sum_{n=1}^{\infty}\left\{A_n\frac{df}{dx}u_n - B_n\left[\frac{d}{dx}\left(\tilde{\mu}\frac{dv_n}{dx}\right) - a^2\tilde{\mu}v_n\right] - C_nN_{Pr}\frac{d}{dx}\left(\frac{d\tilde{\mu}}{dx}\frac{df}{dx}\Theta_n\right)\right\} = \epsilon^{(2)}(x) \quad (32)$$

$$\sum_{n=1}^{\infty}\left\{A_nu_n - C_n\left[\frac{d^2\Theta_n}{dx^2} - a^2\Theta_n\right]\right\} = \epsilon^{(3)}(x) \quad (33)$$

The series given in Equations (28) to (30) will be solutions to Equations (20) to (23) only if the errors $\epsilon^{(1)}(x)$, $\epsilon^{(2)}(x)$, and $\epsilon^{(3)}(x)$ are equal to zero. This condition will be satisfied if the series converge and if each of the errors are orthogonal to a complete set of functions in the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$. The coefficients A_n, B_n, and C_n may then be determined by the requirements

$$\int_{-1/2}^{1/2} \epsilon^{(1)}(x)\, u_m\, dx = 0, \int_{-1/2}^{1/2} \epsilon^{(2)}(x) v_m\, dx = 0,$$

$$\int_{-1/2}^{1/2} \epsilon^{(3)}(x)\, \Theta_m\, dx = 0$$

This leads to an infinite set of linear, homogeneous equations for A_n, B_n, and C_n. A nontrivial solution will exist only if the determinant of the coefficients of A_n, B_n, and C_n is equal to zero. This requirement provides the secular relationship. In practice only a finite number of terms are used in the series expansions for u, v, and Θ. If M terms are used in each series, then a determinant of order $3M$ will result. The exact form used for entries to the determinant is given in reference *17*.

The expansion functions used in this analysis are

$$u_n = x^{n-1}\left(x^2 - \frac{1}{4}\right)^2 \quad (34)$$

$$v_n = \Theta_n = x^{n-1}\left(x^2 - \frac{1}{4}\right) \quad (35)$$

These expansion functions have been used previously to solve problems dealing with stability of flow between concentric cylinders with considerable success. Kurzweg (*18*) used them to solve several small gap problems in hydrodynamic and hydromagnetic stability. Walowit, Tsao, and DiPrima (*5*) have also used them with several finite-gap stability problems.

Although all the resulting integrals can be evaluated analytically, computations are made very tedious by the appearance of quotients of polynomials arising from the present viscosity representation. It was found that for the cases studied, computations could be greatly simplified with little loss in accuracy by expressing the viscosity directly as a polynomial in x. The viscosity data may be approximated fairly accurately with a parabola that passes through the data points at $x = -\frac{1}{2}$, $x = 0$, and $x = \frac{1}{2}$. This does not provide as accurate a representation of the data as the reciprocal quadratic function used in determining the primary velocity profile. However, it was found that the maximum deviation from the data was less than 5% for all the cases studied. Furthermore, no additional error is incurred, since it is not necessary to use this approximation for terms involving viscosity derivatives.

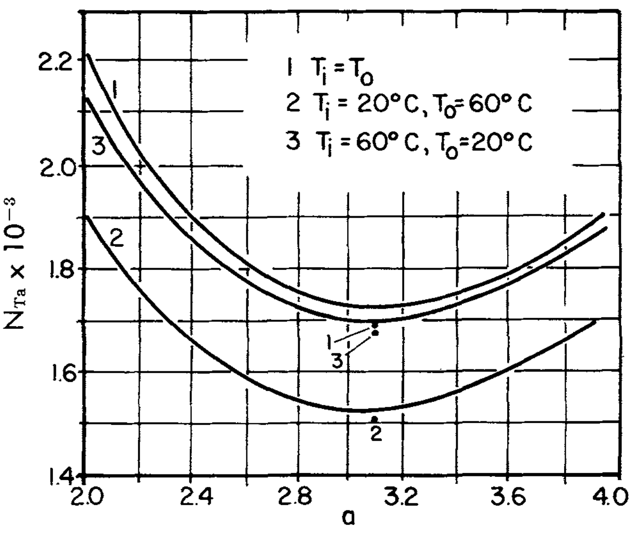

Fig. 1. The variation of N_{Ta} with a for water (points correspond to values of N_{Ta} obtained with three-term approximations).

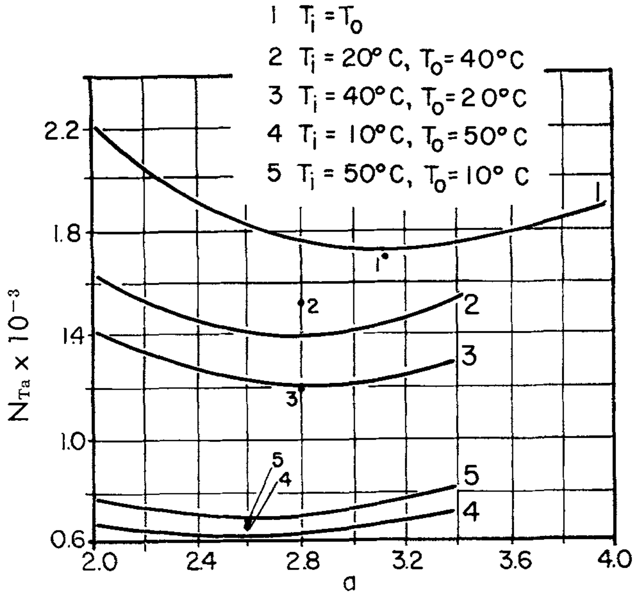

Fig. 2. The variation of N_{Ta} with a for 50% aqueous glycerol (points correspond to values of N_{Ta} obtained with three-term approximations).

Computations were carried out for two- and three-term approximations. The secular determinants were evaluated on an IBM 650 digital computer and the roots were found by trial and error. As a result of the large number of parameters involved, computations were restricted to the case where the outer cylinder is at rest and to contained fluids consisting of water or 50% water-glycerol solutions. Most of the computations were carried out with two-term approximations; the three-term approximations were used primarily as an index of precision.

The Taylor number based on the mean temperature kinematic viscosity was calculated from the modified Taylor number $\tilde{N}_{Ta}$, by the relationship

$$N_{Ta} = \frac{\Omega_i^2 d^3 R_c}{\nu_c^2} = \frac{\left[f\left(\frac{1}{2}\right) - f\left(-\frac{1}{2}\right)\right]^2}{2f\left(\frac{1}{2}\right)} \tilde{N}_{Ta} \quad (36)$$

Figures 1 and 2 show the variation of the Taylor number with the dimensionless disturbance wave number for water and 50% aqueous glycerol solutions calculated with two-term approximations to disturbance amplitudes. The minimum points on these curves determine the critical Taylor numbers at which disturbances begin to amplify and the corresponding critical disturbance wave numbers. Points are also shown corresponding to Taylor numbers computed near the critical points with three-term approximations.

With the exception of the two cases of 50% aqueous glycerol in the presence of a positive radial temperature gradient, results obtained with two-term approximations were found to be within 3% above those obtained with three-term approximations. For the excepted cases, results of the two-term approximations were of the order of 8% below those of three-term approximations.

As an additional check on the precision of the method, solutions to the problem for constant viscosity were compared with those obtained by Lai (*15*). Excellent agreement was obtained over a range of parameters corresponding to cases where $\Omega_o/\Omega_i \geqq 0$.

DISCUSSION

The critical Taylor number varies considerably with the magnitude and direction of the applied temperature gradient as well as the shape of the viscosity profile for the fluids under consideration. The Taylor number evaluated in this analysis is based upon the fluid kinematic viscosity at the average temperature T_c. This viscosity was used since it is the usual way that a viscosity distribution is characterized by a single value in engineering practice. It has been found that for the values of N used the effect on stability due to the interactions of the density gradient with centrifugal acceleration is extremely small. Thus departures of N_{Ta} from 1,695 (the critical Taylor number for the isothermal problem) are due primarily to the effect of viscosity variation. Inspection of Figures 1 and 2 shows that mean temperature viscosity in general does not characterize the influence of a viscosity distribution on stability.

One may define an effective viscosity such that

$$N_{Ta_{eff}} = \frac{\Omega_i^2 d^3 R_i}{\nu^2_{eff}} = 1{,}695$$

For all the cases studied

$$\nu_{min} < \nu_{eff} < \nu_c$$

where ν_{min} is the viscosity at the cylinder having the highest temperature. In general ν_{eff}/ν_c was found to decrease as the Prandtl number increased and as variation in viscosity increased.

The results of previous theoretical analyses show that the effect of radial convection on stability will depend on the direction of heat transfer. In general it has been found that the flow will be destabilized if the temperature increases radially outward and stabilized if the temperature decreases radially outward. The effect of viscosity variation does not result in this type of directional dependence, as seen in Figures 1 and 2. The curves corresponding to water with a cylinder temperature difference of 40°C. show that the lower value of N_{Ta} and consequently the less stable configuration occurs for $T_o > T_i$. On the other hand, for the two cases studied with a 20°C. temperature difference across a 50% glycerol solution, the lower value of N_{Ta} occurs for $T_o < T_i$. For the two cases of 50% glycerol with a 40°C. temperature difference the critical values of N_{Ta} are very nearly the same. It can thus be seen that the direction of heat transfer has no universal effect on stability and particular effects will depend on the overall shape of the viscosity profile of the fluid under consideration.

Previous analyses of the effect of a radial temperature gradient on the stability of Couette flow have neglected both viscosity variations and free convection under the influence of gravity. The effect of gravity on the onset of Taylor vortices still remains to be analyzed. There are a few studies appearing in the present literature concerning the effect of convection on the heat transfer between stationary cylinders. Kraussold (*19*) experimentally correlated heat transfer coefficients with the Rayleigh number defined as

$$R' = \frac{\alpha(T_o - T_i)d^3\sigma}{\nu\kappa}$$

Kraussold found that convective effects were extremely small for Rayleigh numbers less than 10^3. Similar results were later found in experiments with horizontal cylinders carried out by Liu, Mueller, and Landis (*20*) and in the theoretical analysis of Crawford and Lemlich (*21*).

It is thus expected that convective effects due to small temperature differences should be weak when the annular spacing between cylinders is small or the fluid kinematic viscosity is high. Very little is known concerning the influence of convection on the formation of Taylor vortices at high Rayleigh numbers.

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NOTATION

a = disturbance wave number (λd), dimensionless
d = annular spacing
$f(x)$ = function defined by Equation (17)
g = acceleration of gravity
M = number of terms in series expansions
N = dimensionless parameter defined by Equation (26)
N_{Pr} = Prandtl number (ν_c/κ)
R = cylinder radius
r = radial coordinate
R' = Rayleigh number
T = temperature
$\overline{T}$ = primary temperature distribution
T' = temperature disturbance
t = time
N_{Ta} = Taylor number ($\Omega_i^2 d^3 R_c/\nu_c^2$)
$\widetilde{N}_{Ta}$ = modified Taylor number defined by Equation (24)
u = amplitude of radial velocity disturbance, dimensionless
u^* = amplitude of radial velocity disturbance
u' = radial velocity disturbance
u_r = radial velocity component
u_θ = tangential velocity component
u_z = axial velocity component
$\overline{V}$ = primary velocity distribution
v = amplitude of tangential velocity disturbance, dimensionless
v^* = amplitude of tangential velocity disturbance
v' = tangential velocity disturbance
w^* = amplitude of axial velocity disturbance
w' = axial velocity disturbance
x = radial coordinate ($[r - R_c]/d$), dimensionless
z = axial coordinate

Greek Letters

α = bulk coefficient of thermal expansion
γ = constant defined by Equation (21)
Δ = differential operator defined by Equation (10)
Θ = amplitude of temperature disturbance, dimensionless
Θ^* = amplitude of temperature disturbance
κ = thermal diffusivity
λ = disturbance wave number
μ = viscosity

μ = primary viscosity distribution
μ' = viscosity disturbance
$\tilde{\mu}$ = primary viscosity distribution $(\bar{\mu}/\mu_c)$, dimensionless
ν = kinematic viscosity
ρ = density
$\bar{\rho}$ = primary density distribution
ρ' = density disturbance
σ = disturbance growth factor defined by Equation (12)
τ = amplitude of pressure disturbance divided by mean density
Ω = primary angular velocity distribution
$\tilde{\Omega}$ = primary angular velocity distribution (Ω/Ω_c), dimensionless
ξ = dummy radial position variable

Subscripts

c = centerline (points on circle of mean radius)
cr = critical
eff = effective
i = inner cylinder
min = minimum
o = outer cylinder

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The Influence of Diffusivity on Liquid Phase Mass Transfer to the Free Interface in a Stirred Vessel

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Liquid phase mass transfer coefficients were measured in a continuous flow, stirred vessel containing a gas and a liquid phase. Helium, hydrogen, oxygen, argon, and carbon dioxide were desorbed from distilled water into nitrogen at seven different levels of agitation. At low stirring speeds the system was stratified and mass transfer coefficients were proportional to diffusivity raised to a power between 0.5 and 0.6. At higher stirring speeds the interface was broken and corrections for desorption into the entrained bubbles indicated that the mass transfer coefficient at the main free interface was proportional to a higher power of diffusivity. The results are interpreted in the light of a general model considering eddy diffusion and surface renewal effects.

Many industrial processes involve mass transfer between a gas and a liquid. Often the interface between the two phases is relatively free to move about in space and is deformable; such is the case, for example, in plate distillation columns and for the main interface in agitated vessels. This characteristic has led to several different approaches to the description of liquid phase mass transfer near a free gas-liquid interface. Well-known theories include the film model of Whitman and Lewis (*30*), the penetration and surface renewal models of Higbie (*14*) and Danckwerts (*4*), and the postulate of turbulence-controlled mass transfer put forward by Kishinevsky (*20, 21*) and Kafarov (*18*). There are also "combination" theories, such as the film-penetration model (*7, 26*).

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